

2. EULERIAN AND HAMILTONIAN GRAPHS

§2.1. Cycles

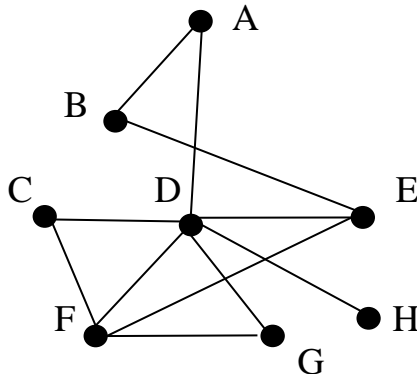
Recall that a **walk** in a graph is a sequence of edges

$$E_1, E_2, \dots, E_m$$

where, for $i = 1, \dots, m - 1$, the end of E_i is the same vertex as the start of E_{i+1} .

A **path** is a walk where no edge is repeated and a cycle is a path which starts and finishes at the same vertex. The **length** of the walk is its number of edges.

Example 1: How many cycles of length 4 are there in the following graph.



Solution: By inspection the paths of length 4 are:
ABEDA, CDGFC, CDEFC and their reverses
ADEBA, CFGDC and CFEDC.

Hence there are 6 of them.

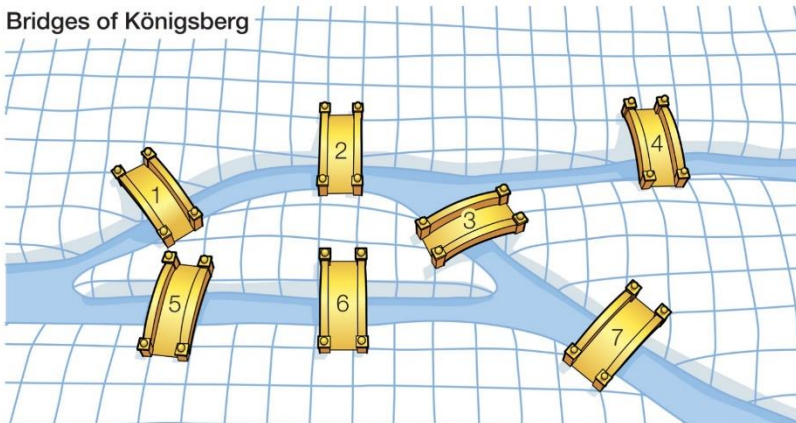
Suppose we wanted to write a computer program to answer such questions. In that case we'd input the graph as an adjacency matrix. Here it would be an 8×8 matrix, A , of 0's and 1's where $a_{ij} = 1$ if and only if there is an edge from i to j .

The components on the diagonal of A^4 would give the number of *walks* from a vertex to itself and We would have to subtract off those walks that repeat edges.

A walk of length 4 that repeats edges could also have the form $XYXYX$, that is a path of length 2 repeated.

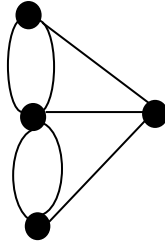
§2.2. Eulerian Graphs

The Swiss mathematician Leonard Euler (pronounced “Oiler”) once asked whether it was possible to go for a



walk around the city of Königsburg, crossing each of the seven bridges exactly once.

There are 4 pieces of land and 7 bridges. We can represent them by a graph with 4 vertices and 7 edges as follows.

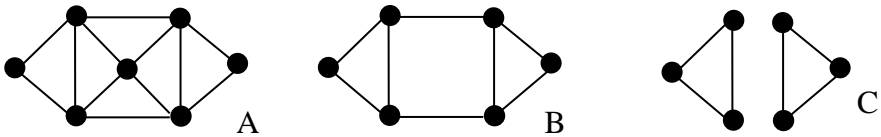


This is a graph with multiple edges and we have chosen to focus on graphs with at most one edge connecting any two vertices. However the reason there is no such path extends to graphs in general, whether or not there are multiple edges.

We must enter each vertex and leave each vertex so the degree of each vertex must be even. Euler didn't make it clear whether one had to return to the starting point. If not, the degree of these two vertices could be odd. But the graph in question has all four vertices with odd degree. The only way we could achieve the goal would be to take a helicopter ride at some point. If we had to return to where we started, two helicopter rides would be necessary. This is not what Euler had in mind.

An **Eulerian cycle** in a graph (undirected with no multiple edges) is one that passes along every edge exactly once. An **Eulerian graph** is one that has an Eulerian cycle. (Remember to pronounce Eulerian as “Oil-air-ian”.)

Example 2: Which of the following graphs are Eulerian?



Solution: A is Eulerian, but not B or C. The third graph is clearly not Eulerian because it’s not connected. But being connected is clearly not enough.

Theorem 1: An undirected graph is Eulerian if and only if it is connected, and every vertex has even degree.

Proof: Suppose G is Eulerian. Clearly G must be connected. Moreover, since we must enter and exit every vertex along different edges the degree of every vertex must be even.

Now suppose that G is connected and every vertex has even degree. We prove the theorem by induction on the number of edges. Suppose G has E edges and the theorem is true for graphs with fewer than E edges.

G clearly has a cycle. Simply start at some point and move to another vertex. Since every vertex has degree at least 2 we can continue the path until we repeat a

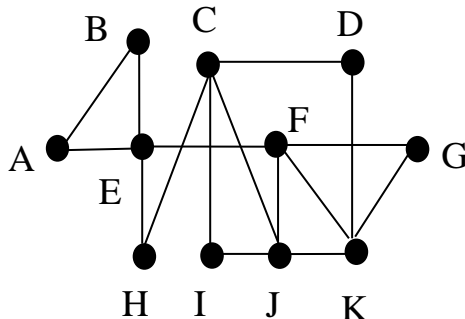
vertex. That portion of the path between the two visits to that vertex will be a cycle.

Remove the edges of this cycle and remove any vertices that are now isolated (these would have had degree 2 in the original graph), The resulting graph will have every vertex of even degree (we have removed 2 edges from each vertex visited in this cycle). But it may no longer be connected. Let C_1, C_2, \dots, C_k be these components.

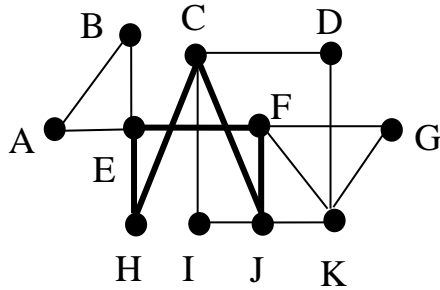
However each component is connected and will have every one of its vertices with even degree. Moreover these components will have fewer edges than G . Hence there is an Eulerian cycle in each of these components.

Now, start at one of the vertices in the original cycle. This vertex will be in one of the C_i . Now trace out the Eulerian cycle in this component. Continue around the original cycle. However, every time you enter a new C_i for the first time trace out the corresponding Eulerian cycle. Finally you'll return to your starting point. This will be an Eulerian cycle for the graph G .

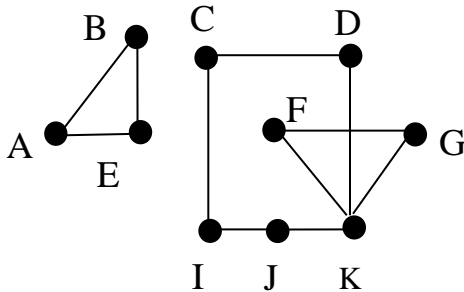
Example 3: Suppose G is the graph:



Start at A and travel the path AEHCJFE. Having repeated E we find the cycle EHCJFE.



Now remove this cycle.

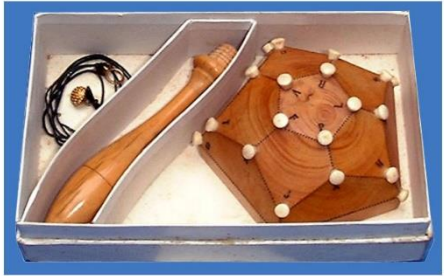


There are 2 components. Each has an Eulerian cycle, for example, ABE and CDKGFKJIC.

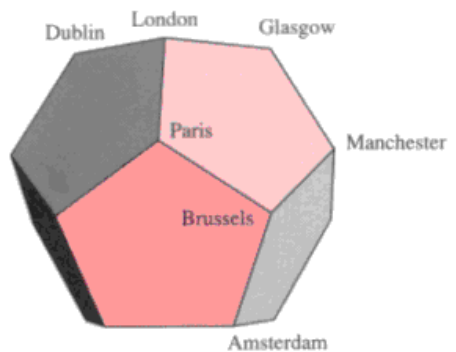
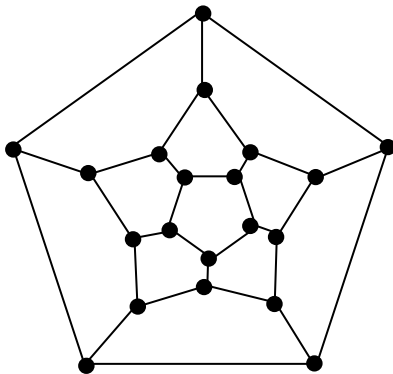
Combining these with the original cycle, as described in the proof of Theorem 2, we get the Eulerian cycle EABE H CDKGFKJIC JFE.

§2.3. Hamiltonian Graphs

In 1857 the mathematician Sir William Hamilton created a puzzle that, in its day, was almost as popular as the Rubik's Cube. It was a wooden dodecahedron in which the 20 vertices were labelled with the name of a city. The problem was to make a world tour, starting and finishing at the same city and passing along the edges, so that each other city is visited exactly once.



We don't need a solid dodecahedron to solve the puzzle. The edges form a graph on a sphere that becomes a planar graph when we flatten it out.



A **Hamiltonian path** in an undirected graph is a path that passes through each vertex exactly once. It clearly cannot

pass along each edge more than once, but some edges will not occur.

A **Hamiltonian cycle** in an undirected graph is a cycle that passes through every vertex exactly once. (Of course it will return to the vertex where it began.)

A **Hamiltonian graph** is an undirected graph that has a Hamiltonian cycle. Clearly a Hamiltonian graph must be connected.

Hamilton's Dodecahedron Puzzle is equivalent to finding a Hamiltonian cycle in the above graph.

Theorem 2 (ORE): If $\deg(X) + \deg(Y) \geq n$ whenever X is not adjacent to Y , in an undirected graph G with n vertices, then G is Hamiltonian.

Proof: Suppose G satisfies these conditions but is not Hamiltonian. Add extra edges until you reach a graph H such that H is not Hamiltonian but adding one extra edge UV makes a Hamiltonian graph K . Clearly this edge must be included in every Hamiltonian cycle of K .

Removing this edge will give a Hamiltonian path:

$$V_1 V_2 V_3 \dots V_{n-1} V_n \text{ in } H.$$

Everything that follows refers to the graph H .

Since G is a subgraph of H the assumption of the theorem carries across to H . This is because two vertices being non-adjacent in H implies that they are non-adjacent in G

and the degree of every vertex in H is greater than or equal to its degree in G . Also, since H has exactly the same vertices as G , H has n vertices.

Let $S = \{i \mid V_i \text{ is adjacent to } V_1\}$.

Then $\#S = \deg(V_1)$. Clearly $1 \notin S$.

Moreover $n \notin S$, for if V_n is adjacent to V_1 then adding this to the Hamiltonian path would give a Hamiltonian cycle.

Let $T = \{i \mid i + 1 \in S\}$.

Then $\#T = \#S = \deg(V_1)$. Also $V_n \notin T$.

Let $R = \{i \mid V_i \text{ is adjacent to } V_n\}$.

Then $\#R = \deg(V_n)$.

Clearly $V_n \notin R$.

Since $n \notin S$, V_1 and V_n are not adjacent and so:

$$\deg(V_1) + \deg(V_n) \geq n.$$

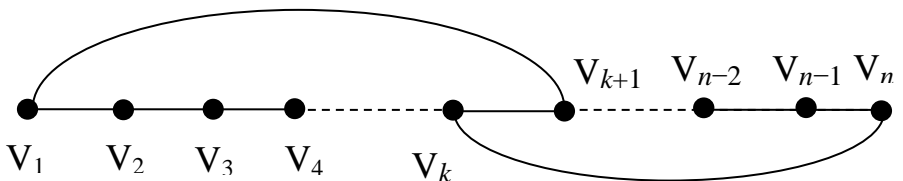
Hence $\#T + \#R \geq n$.

Since $\#(T \cup R) \leq n - 1$, $\#(T \cap R) \geq 1$.

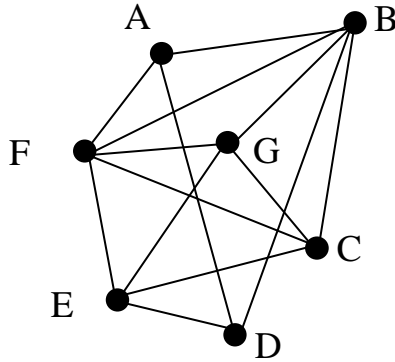
Let $V_k \in T \cap R$.

Then $V_k V_n$ and $V_{k+1} V_1$ are edges in H .

Then $V_1 V_2, V_3 \dots V_{k-1} V_k V_n V_{n-1} V_{n-2} \dots V_{k+2} V_{k+1} V_1$ is a Hamiltonian cycle in H , a contradiction. 🙅😊



Example 4: Let G be the following graph.



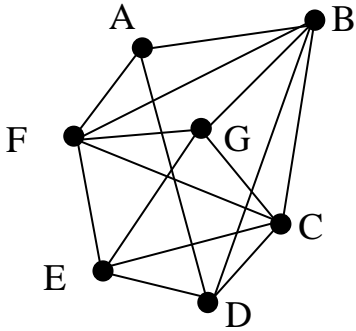
The number of vertices is $n = 7$. The degrees of the vertices are as follows:

| A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|
| 3 | 5 | 4 | 3 | 4 | 4 | 5 |

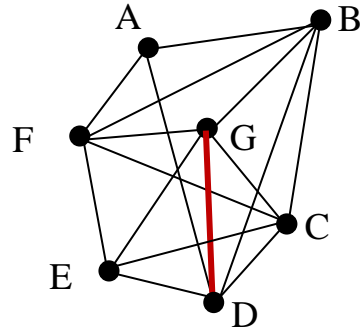
The pairs of non-adjacent vertices are: AC, AE, AG, BE and CD

At first glance it might appear that G is not Hamiltonian. Suppose this is the case. Suppose that when we add the edge CD it is still not Hamiltonian. But adding the edge GD as well the graph is clearly Hamiltonian. So in terms of the notation of the above theorem:

H is:



and K is:



The cycle ABCGDEFA is a Hamiltonian cycle in K and DEFABCGD is a Hamiltonian path in H.

$V_1 = D, V_2 = E, V_3 = F, V_4 = A, V_5 = B, V_6 = C, V_7 = G.$

$S = \{2, 4, 5, 6, 7\}, T = \{1, 3, 4, 5, 6\}, R = \{1, 2, 3, 6\}.$

$\#T = 5 = \#S = \deg(D) = 5. \#R = \deg(G) = 5.$

So $T \cap R = \{1, 3, 6\}.$

Suppose we choose $k = 1$, so $k + 1 = 2.$

Then we get the Eulerian cycle D GCBAFE D in H.

Suppose we choose $k = 3$, so $k + 1 = 4.$

Then we get the Eulerian cycle DEF GCBA D in H.

Suppose we choose $k = 6$, so $k + 1 = 7.$

Then we get the Eulerian cycle DEFABC G D in H.

These are not Hamiltonian cycles for G, but DEFGCBAD is. So G is indeed Hamiltonian, as it ought to be since it satisfies the hypothesis of Theorem 2.

